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p-Gerbes and Extended Objects in String Theory

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Abstract

p-Gerbes are a generalization of bundles that have $(p+2)$-form field strengths. We develop their properties and use them to show that every theory of p-gerbes can be reinterpreted as a gauge theory containing p-dimensional extended objects. In particular, we show that every closed $(p+2)$-form with integer cohomology is the field strength for a gerbe, and that every p-gerbe is equivalent to a bundle with connection on the space of p-dimensional submanifolds of the original space. We also show that p-gerbes are equivalent to sheaves of $(p-1)$-gerbes, and use this to define a K-theory of gerbes. This K-theory classifies the charges of $(p+1)$-form connections in the same way that bundle K-theory classifies 1-form connections.

1 Introduction

p-Gerbes are a generalization of fiber bundles which have higher form connections. For $p = 0$, they are bundles. The case $p = 1$ was introduced by Giraud [1] and refined by Brylinski [2] as a tool to study the properties of 3-manifolds. A good introduction to their properties was given by Hitchin [3]. The case $p = 2$ was developed in [4] in order to study higher cohomology classes in gauge theories.

Gerbes are valuable because they provide a geometric way to unify the properties of p-form fields with gauge symmetries. We will begin by studying the detailed properties of these objects. We will show that every closed $(p+2)$-form with integral cohomology is the field strength of a p-gerbe, and that p-gerbes are equivalent to bundles with connection on the space of smooth p-dimensional submanifolds of the original base space. This means that any theory of higher forms implicitly is a theory of extended objects; at the end of this paper, we will make this relationship explicit, showing how the higher-form fields can be replaced by the integrals of 1-forms over p-dimensional internal spaces.

It will also be useful to derive a better topological and geometric picture of gerbes. To this end, we study their local properties, and show that they have sections; in fact, p-gerbes are equivalent to sheaves of $(p-1)$-gerbes. This allows us to develop a K-theory of gerbes analogous to that of bundles, which (for similar reasons) classifies the higher-form charges of extended objects in string theory. The results are consistent with the known NS B-field charges in type II string theory.

The paper is laid out as follows. In section 2, we define p-gerbes and introduce three equivalent pictures thereof:

- Čech language: A p-gerbe on a manifold X over a Lie group K can be thought of as an open cover of the space along with K-valued transition functions on $(p+2)$-fold intersections. This language contains the underlying definition of a gerbe and is useful for computations.

- de Rham language: p-gerbes have $(p+2)$-form field strengths and $(p+1)$-form connections. We show that every closed $(p+2)$-form on a manifold with integral Chern class is the field strength of some p-gerbe. These gerbes have a gauge symmetry of the form $B \to B + dA$, where B is the connection and A is an arbitrary p-form.

- Loop language: p-gerbes implement gauge symmetries on the space of p-loops

in the same way that bundles (which are 0-gerbes) implement gauge symmetries on the original space. In particular, gauge symmetries involving closed strings are naturally associated with 1-gerbes, and the Neveu-Schwarz tensor field can be interpreted as the connection associated with such a symmetry.

In section three, we study their local structure and define a fourth picture:

- Sheaf language: At least for Abelian K, a p-gerbe is a sheaf of $(p-1)$-gerbes. An example (due to Hitchin) is a space which does not support a Spin^c structure for topological reasons; such a space can be covered with open sets, on each of which such a structure is defined. The combination of all such sets and their transition functions forms a 1-gerbe, since each structure is a bundle.[1]

In section 4, we use the Čech and sheaf pictures to define the K-theory of gerbes, and show that it behaves very similarly to that of bundles. Finally, we return to the question of how extended objects emerge from gerbe theories and show the explicit correspondence.

As this work was being prepared for publication, we became aware of related work by Ekstrand [5] which develops the Čech and de Rham pictures in detail. The work of Freund and Nepomechie [6] has also been brought to our attention, in which the relationship between $(p+1)$-forms over $U(1)$ and connections on p-loop spaces was developed.

2 Transition functions, Connections, and Loops: An overview of gerbes

We begin with a definition. Let X be a manifold and K be a Lie group. A p-gerbe ξ on X over K is a pair (U, g), where U_α is a good open cover (one whose intersections are contractible) of X, and $g_{\alpha_1\alpha_2\cdots\alpha_{p+2}}$ is a collection of functions $U_{\alpha_1\cdots\alpha_{p+2}} \equiv U_{\alpha_1} \cap \cdots \cap U_{\alpha_{p+2}} \to K$ on every $(p+2)$-fold intersection satisfying the *inversion condition*

$$g_{\alpha_1\cdots\alpha_i\cdots\alpha_j\cdots\alpha_{p+2}} = g^{-1}_{\alpha_1\cdots\alpha_j\cdots\alpha_i\cdots\alpha_{p+2}} \tag{1}$$

and the *cocycle condition* on $(p+3)$-fold intersections

[1] There is also a fifth language, that of sheaves of groupoids, in terms of which (1-)gerbes were originally introduced; we will not discuss this here, but refer the interested reader to [2].

$$(\delta g)_{\alpha_1\cdots\alpha_{p+3}} = g_{\alpha_2\cdots\alpha_{p+3}} g^{-1}_{\alpha_1\alpha_3\cdots\alpha_{p+3}} g_{\alpha_1\alpha_2\alpha_4\cdots\alpha_{p+3}} \cdots g^{(-1)\cdot p}_{\alpha_1\cdots\alpha_{p+2}} = 1 \ . \tag{2}$$

For $p = 0$, this reduces to the definition of a K-bundle. The definition (1,2) is based on transition functions and so is somewhat hard to visualize; for one thing, for $p > 0$ a gerbe is not a manifold. (As is the total space of a bundle) Later on we will see that gerbes nonetheless have a well-defined notion of section, and in fact a p-gerbe is a sheaf of $(p-1)$-gerbes.

We will denote the set of all p-gerbes on a given manifold and group by $G_p(X, K)$. For consistency in our recursive definitions, we will also denote by $G_{-1}(X, K)$ the set $C(X, K)$ of continuous functions from X to K, and by $G_{-2}(X, K)$ the group K itself. Since g is a $(p+2)$-cocycle, a gerbe $\xi \in G_p(X, K)$ is naturally topologically classified by the Čech cohomology group $H^{p+2}(\xi) \equiv H^{p+2}(g \in C^{p+2}(X, K))$. This is clearly invariant under continuous deformations (homomorphisms) of ξ. Similarly we can naturally define pullbacks $\omega^\star\xi$ of a gerbe to a submanifold $\omega \subset X$ and tensor products $\xi \otimes \xi'$ of gerbes. This construction is the Čech picture of gerbes.

We now define the de Rham (connection) picture. Define the $\mathfrak{alg}\ K$-valued $(p+2)$-cochain

$$A^{(0)}_{\alpha_1\cdots\alpha_{p+2}} = \log g_{\alpha_1\cdots\alpha_{p+2}} \ . \tag{3}$$

Since g is a cocycle, we know that $\delta A^{(0)} = g^{-1}\delta g = 1$, and so $\delta dA^{(0)} = 0$. This means that using the Poincaré lemma we can define a 1-form valued $(p+1)$-cochain $A^{(1)}_{\alpha_1\cdots\alpha_{p+1}}$ satisfying

$$\left(\delta A^{(1)}\right)_{\alpha_1\cdots\alpha_{p+2}} - dA^{(0)}_{\alpha_1\cdots\alpha_{p+2}} = 0 \tag{4}$$

on every $(p+2)$-fold intersection. Since $\delta dA^{(1)} = ddA^{(0)} = 0$, we can repeat this process, defining a sequence of n-form valued $(p-n+2)$-cochains $A^{(n)}$ by

$$\delta A^{(n+1)} - dA^{(n)} = 0 \ . \tag{5}$$

Such an iterated use of the Poincaré lemma is simply the standard relationship of Čech to de Rham cohomology. This sequence ends when we define the $(p+1)$-form $A^{(p+1)}_\alpha$ on every open set U_α, which by the Poincaré lemma satisfies

$$dA^{(p+1)}_\alpha = A^{(p+2)}\big|_{U_\alpha} \ . \tag{6}$$

$H \equiv A^{(p+2)}$ is a globally defined $(p+2)$-form which is the noncovariant field strength

of the gerbe.[2] $B \equiv A^{(p+1)}$ is its $(p+1)$-form connection, which is defined on each coordinate patch.

This process can be reversed, as well; if we are given a closed $(p+2)$-form H and a contractible cover U_α, then by the Poincaré lemma there are $(p+1)$-forms B on each U_α such that $H|_{U_\alpha} = dB_\alpha$. Then on any $U_{\alpha\beta}$, $\delta dB = dB_\alpha - dB_\beta = 0$, and so we can define p-forms $A^{(p)}$ such that $\delta B = dA^{(p)}$, and so forth until we come again to the 0-forms $A^{(0)}$. Then if H defines an integral class in $H^{p+2}_{dR}(X, K)$ the exponential of $A^{(0)}$ is well-defined and so there are cochains $g = exp\ A^{(0)}$, which therefore form a gerbe. So to each closed $(p+2)$-form defining an integral class corresponds a p-gerbe. This alows us to pass freely between the Čech and de Rham pictures.

There is however an ambiguity in the descending construction. While we know that $H = dB$, it is possible to shift B by any closed $(p+1)$-form and maintain this. Therefore p-gerbes have a gauge symmetry generated by a p-form:

$$B \to B + d\xi^{(p)}\ . \tag{7}$$

Similarly there are lower gauge symmetries for each $A^{(n)}$, generated by $(n-1)$-forms. Once these gauge symmetries are equivalenced out, we find that

Theorem 2.1: The set of p-gerbes is given by the set of closed $(p+2)$-forms (field strengths) with integral de Rham cohomology class in H^{p+2}.

The gauge symmetry (7) is familiar from the NS B-field in string theory. By this theorem, we can interpret this field as a connection on a 1-gerbe. This agrees with the result of [12] that gerbes describe the B-field in massive IIA supergravity. We will see the geometric interpretation of this relationship below.

At first it may seem odd to define a connection which is not associated with an obvious covariant derivative. A way to define such a derivative is suggested by the result of [2] that 1-gerbes are equivalent to K-bundles over the loop space ΩX,[3] and by a theorem due to Getzler, Jones and Petrack [13] that the set of k-forms on ΩX is isomorphic to the set of 1-cochains of k-forms on X.

Let us begin with this theorem. If we iterate it, defining the pth loop space by $\Omega^p X \equiv \Omega\Omega^{p-1}X$, k-forms on $\Omega^p X$ are isomorphic to p-cochains of k-forms on X, which (by the usual exchange of Čech and spacetime indices) are $(k+p)$-forms on

[2]We have here used partial derivatives, and so this field strength is not the one ordinarily used in physics for K non-abelian. In particular, it will not have the usual gauge invariance. Below we will define a covariant field strength which remedies this.

[3]Defined to be the set of embeddings of $S^1 \to X$, modulo reparametrizations of the S^1.

X. Then it is natural to suspect that the $(p+1)$-form B defined on each U_α can be interpreted as a 1-form on the loop space $\Omega^p X$. Using this, we could define a natural action of a p-gerbe on $\Omega^p X$ by means of a covariant derivative

$$\nabla = d + B \tag{8}$$

which describes how functions $f(\omega) : \Omega^p X \to K$ transform under infinitesimal deformations $\omega \to \omega + \delta\omega$, where $\delta\omega \sim \omega \times [0,1]$;

$$f(\omega) \to f(\omega + \delta\omega) = \exp[\delta\omega \cdot \nabla] f(\omega) \ . \tag{9}$$

The dot product of the $(p+1)$-cochain $\delta\omega$ and B is given by the usual de Rham product

$$\delta\omega \cdot B = \int_{\delta\omega} B \ . \tag{10}$$

The curvature of this covariant derivative is $[\nabla, \nabla]$, a 2-form on $\Omega^p X$ which is therefore a $(p+2)$-form on X. This is the covariant generalization of our ordinary field strength. Note that this definition is meaningful even when K is non-Abelian, and so gives a natural way to define Bianchi identities for higher gerbes. However, although covariant derivatives can relate the connection to the field strength, there is no natural way to define the lower forms $A^{(n)}$ in this way, so one can only pass from the loop picture to the Čech picture in terms of partial derivatives. (The noncovariant field strength is, however, still defined and useful in the non-Abelian case)

The one subtlety that might obstruct the definition (8) is that a given loop ω, or its variation $\delta\omega$, may overlap multiple U_α and so no B could be defined on the entire loop. To show that this is not the case, we will need a result from sections 3 and 4 that gerbes with trivial H^{p+2} have a global section and so are equivalent to $(p-1)$-gerbes on the space. Since $\delta\omega$ is $(p+1)$-dimensional, $H^{p+2}(\delta\omega) = 0$, and so the pullback $\delta\omega^\star \xi$ for any $\xi \in G_p(X, K)$ is trivial. This means that the restriction of ξ to $\delta\omega$ has a global section, and so $\xi \in G_{p-1}(\delta\omega, K)$. Therefore the covariant derivative (8) can be defined for any gerbe over the appropriate loop space. Similarly, any connection on $\Omega^p X$ can be converted to a collection of $(p+1)$-forms on every open set of X, which by theorem 2.1 defines a p-gerbe. We therefore have a well-defined "loop picture" of gerbes, and

Proposition 2.2: $G_p(X, K) \cong G_0(\Omega^p X, K)$. ($p$-gerbes on X are bundles on $\Omega^p X$)

Since these bundles have a connection, we may interpret this to say that p-gerbes implement gauge symmetries on p-fold loop spaces in the same way that bundles implement gauge symmetries on points. Combining this with the de Rham picture, this means that $(p+1)$-form connections can be interpreted as connections on the space of p-loops.

We have not, in this discussion, used the fact that the ω are actually loops; we may naturally consider what would happen if instead $\omega \in M^p X$, the space of smooth p-manifolds smoothly embedded in X. (This is the p-dimensional analogue of the unfixed path space) One would expect that the type of gerbe needed to implement gauge transformations on a manifold ω should not change under small deformations of ω such as "smearing" over an interval. Specifically, one expects that if $\omega \in M^p X$ is contractible to $\eta \in M^q X$, with $q < p$, then a q-gerbe should suffice to define gauge symmetries on ω. This can easily be shown. Let $\xi \in G_p(X, K)$; then $\delta\omega^\star\xi$ is trivial, so $\xi \in G_{p-1}(\delta\omega, K)$. Since $\delta\omega \sim \omega \times [0,1] \sim \eta \times [0,1] \sim \delta\eta$, we know that $\xi \sim \xi' \in G_{p-1}(\delta\eta, K)$. But since dim $\delta\eta = q+1$, $H^k(\xi') = 0$ for $k > q+1$, and since cohomology classes are invariant under homomorphism, $H^k(\xi) = 0$ for $k > q+1$. Therefore $\xi \in G_{q-1}(\delta\omega, K)$, and so

Theorem 2.3: Let $M^p_q X = \{\omega \in M^p X : \inf\{\dim \eta : \eta \sim \omega\} = q\}$. Then every element of $G_q(X, K)$ defines a connection on $M^p_q X$.

Equivalently, decompose $\bigoplus_p M^p X = \bigoplus_{p,q} M^p_q X = \bigoplus_q M_q X$, where $M_q = \bigoplus_p M^p_q X$ is the set of all submanifolds of X which can be contracted down to q dimensions. Then $G_q(X, K) \cong G_0(M_q X, K)$.

That is, q-gerbes define connections on the space of curves homomorphic to q-loops in X.[4] This means that connections on the space of open strings (embeddings of $[0,1] \to X$, where X is spacetime) take values in $G_0(X, K)$, but connections on closed strings take values in $G_1(X, K)$ since circles cannot be contracted to a point. We can therefore also physically interpret the NS tensor field $B^{\mu\nu}$ as a connection on a 1-gerbe which implements a $U(1)$ gauge symmetry on the space of closed strings. This gauge symmetry is identical to the symmetry which transforms the vector field A^μ in open string theories in the absence of background D-branes.

We now have three pictures of gerbes: a Čech picture, given by open covers and transition functions; a de Rham picture, given by a $(p+2)$-form field strength

[4]This generalizes the result of [6], which in our language states that $G_q(X, U(1)) \cong G_0(M^q X, U(1))$.

with $(p+1)$-form connections and p-form gauge symmetries; and a loop picture, with bundles on spaces of p-loops. It will be useful for us to introduce still a fourth picture, which will describe gerbes in terms of their local (section) structure. This picture will both give additional intuition as to the nature of gerbes and aid in calculations, especially in the definition of a K-theory of gerbes.

3 Sections of gerbes

In this section we will restrict ourselves to the case of K abelian. We wish to determine what a gerbe looks like "locally," i.e. the analogue for gerbes of sections of bundles. To do this we will first define an auxilliary structure called a pregerbe, which is identical to a gerbe except it does not satisfy the cocycle condition (2). Instead we define the variation $\delta\xi$ of a pregerbe ξ to be the set of coboundaries of the transition functions of the pregerbe;

$$h_{\alpha_1\cdots\alpha_{p+3}} \equiv (\delta g)_{\alpha_1\cdots\alpha_{p+3}} \; . \tag{11}$$

A pregerbe is a gerbe if all elements of $\delta\xi$ are unity. By the Poincaré lemma, $\delta h = \delta\delta g = 1$, and so the variation of any pregerbe is a gerbe. We denote the class of p-pregerbes by $PG_p(X, K)$.

We also define a notion of equivalence for two pregerbes on the same manifold. For $\xi = (U, g)$ and $\xi' = (U', g')$, we define the mutual refinement $U \cap U'$ of the two covers to be the set of all intersections of elements of U with elements of U'. Clearly both ξ and ξ' have a natural extension to this mutual refinement. Then we say that $\xi \cong \xi'$ if $\delta\xi = \delta\xi'$ on each $(p+3)$-fold intersection in $U \cap U'$. (i.e., if their variations define the same gerbe)

We begin by proving a simple but useful lemma. We define the difference of two pregerbes $\xi = (U, g)$ and $\chi = (U', g')$ to be

$$\chi - \xi = (U \cap U', g'_{\alpha_1\cdots\alpha_{p+2}} g^{-1}_{\alpha_1\cdots\alpha_{p+2}}) \; . \tag{12}$$

Then

Lemma 3.1: $\delta(\chi - \xi) = 1$ iff $\chi \cong \xi$. (The difference of two equivalent pregerbes is a gerbe)

Proof. This follows from direct evaluation of the variation. On a $(p+3)$-fold intersection $U_{\alpha_1\cdots\alpha_{p+3}}$,

$$\delta(\chi - \xi) \;\; = \;\; (g'g^{-1})_{\alpha_2\cdots\alpha_{p+3}} (g'g^{-1})^{-1}_{\alpha_1\alpha_3\cdots\alpha_{p+3}} \cdots (g'g^{-1})^{(-1)^p}_{\alpha_1\cdots\alpha_{p+2}}$$

$$
\begin{aligned}
&= g'_{\alpha_2\cdots\alpha_{p+3}} g'^{-1}_{\alpha_1\alpha_2\cdots\alpha_{p+3}} \cdots g'^{(-1)^p}_{\alpha_1\cdots\alpha_{p+2}} g^{(-1)^{(p+1)}}_{\alpha_1\cdots\alpha_{p+2}} g_{\alpha_1\alpha_3\cdots\alpha_{p+3}} \cdots g^{-1}_{\alpha_2\cdots\alpha_{p+3}} \\
&= \delta g'_{\alpha_1\cdots\alpha_{p+3}} (\delta g_{\alpha_1\cdots\alpha_{p+3}})^{-1}
\end{aligned}
$$

Which is equal to unity iff the two variations are equal on all intersections.

This associates a unique p-gerbe with each equivalence class of p-pregerbes. Likewise every gerbe can be written as the variation of some pregerbe; thus

Lemma 3.2: The set of equivalence classes in PG_p is isomorphic to G_p.

We can describe the local structure of gerbes in terms of pregerbes. We say that a local trivialization of a gerbe $\xi = (U, g)$ on a $(p+1)$-fold intersection $U_{\alpha_1\cdots\alpha_{p+1}}$ is a realization of the cocycle g as a coboundary, i.e. a collection of functions $f_{\alpha_1\cdots\alpha_{p+1}}$ such that

$$
g_{\alpha_1\cdots\alpha_{p+2}} = (\delta f)_{\alpha_1\cdots\alpha_{p+2}} = f_{\alpha_2\cdots\alpha_{p+2}} f^{-1}_{\alpha_1\alpha_3\cdots\alpha_{p+2}} \cdots f^{(-1)^{(p-1)}}_{\alpha_1\cdots\alpha_{p+1}} \tag{13}
$$

The $f^{(\alpha)}$ are simply a $(p-1)$-pregerbe on $U_{\alpha_1\cdots\alpha_{p+1}}$ whose variation is ξ. Clearly, such trivializations are not unique; the set of all local trivializations of a given ξ is an equivalence class of pregerbes on $U_{\alpha_1\cdots\alpha_{p+1}}$. A trivialization of ξ is a collection of local trivializations on every such intersection; a global trivialization is a single local trivialization defined simultaneously over all of X. We will see that the condition for a global trivialization to exist is that $H^{p+2}(\xi)$ is trivial.

Trivializations, however, can always be constructed. One trivialization of particular interest is given by choosing on each U_α

$$
f^{(\alpha)}_{\beta_1\cdots\beta_{p+1}} \equiv g_{\alpha\beta_1\cdots\beta_{p+1}} \tag{14}
$$

This is then defined on each $(p+1)$-fold intersection, and forms a $(p-1)$-pregerbe on each U_α, whose variation is the restriction of the original gerbe to that set. The collection of all such $f^{(\alpha)}$ forms a trivialization valid on each U_α, since

$$
\begin{aligned}
\left(\delta f^{(\alpha)}\right)_{\beta_1\cdots\beta_{p+2}} &= f^{(\alpha)}_{\beta_2\cdots\beta_{p+2}} \cdots f^{(\alpha)[(-1)^{(p-1)}]}_{\beta_1\cdots\beta_{p+1}} \\
&= g_{\alpha\beta_2\cdots\beta_{p+2}} \cdots g^{(-1)^{(p-1)}}_{\alpha\beta_1\cdots\beta_{p+1}} \\
&= g_{\beta_1\cdots\beta_{p+2}} \, (\delta g)_{\alpha\beta_1\cdots\beta_{p+2}} \\
&= g_{\beta_1\cdots\beta_{p+2}}
\end{aligned} \tag{15}
$$

We call each $f^{(\alpha)}$ a section of ξ on U_α.

This term is justified by showing that the set of such $f^{(\alpha)}$ forms a sheaf of pregerbes, which (since each $f^{(\alpha)}$ is a representative of an equivalence class of pregerbes)

makes the set of local trivializations of ξ into a sheaf of equivalence classes of pregerbes. To do this, we note that on each U_α we have defined a collection of functions $f^{(\alpha)}_{\beta_1\cdots\beta_{p+1}}$ which map the intersection $U_\alpha \cap U_{\beta_1\cdots\beta_{p+1}}$ to K. If the $f^{(\alpha)}$ form a sheaf, there must be transition functions for each of these functions on intersections $U_\alpha \cap U_{\alpha'}$. These follow from the cocycle condition on ξ;

$$(\delta g)_{\alpha\alpha'\beta_1\cdots\beta_{p+1}} = g_{\alpha'\beta_1\cdots\beta_{p+1}} g^{-1}_{\alpha\beta_1\cdots\beta_{p+1}} g_{\alpha\alpha'\beta_2\cdots\beta_{p+1}} \cdots g^{(-1)^p}_{\alpha\alpha'\beta_1\cdots\beta_p} = 1 \tag{16}$$

and so

$$(f^{(\alpha)})_{\beta_1\cdots\beta_{p+1}} = (\phi_{\alpha\alpha'})_{\beta_1\cdots\beta_{p+1}} (f^{(\alpha')})_{\beta_1\cdots\beta_{p+1}} \tag{17}$$

where

$$\phi_{\alpha\alpha'} = g_{\alpha\alpha'\beta_2\cdots\beta_{p+1}} g^{(-1)}_{\alpha\alpha'\beta_1\beta_3\cdots\beta_{p+1}} \cdots g^{(-1)^p}_{\alpha\alpha'\beta_1\cdots\beta_p} \; . \tag{18}$$

The $\phi_{\alpha\alpha'}$ clearly satisfy the inversion condition $\phi_{\alpha\alpha'}\phi_{\alpha'\alpha} = 1$; they also satisfy the cocycle condition

$$\phi_{\alpha_1\alpha_2}\phi_{\alpha_2\alpha_3}\phi_{\alpha_3\alpha_1} = f^{(\alpha_1)} f^{(\alpha_2)-1} f^{(\alpha_2)} f^{(\alpha_3)-1} f^{(\alpha_3)} f^{(\alpha_1)-1} = 1 \tag{19}$$

and so they indeed are the transition functions on a sheaf. Therefore (since on each U_α this trivialization is a representative of the equivalence class of all local trivializations) we see that

Lemma 3.3: G_p is isomorphic to the set of equivalence classes of sheaves of PG_{p-1}.

It then follows from lemmas 3.2 and 3.3 that

Theorem 3.4: The set of p–gerbes is isomorphic to the set of sheaves of $(p-1)$–gerbes.

Proof. Each element of G_p is isomorphic to a equivalence class of sheaves of PG_{p-1}, which is isomorphic to a sheaf of equivalence classes of PG_{p-1}, which is a sheaf of $(p-1)$–gerbes.

This allows us to think of gerbes as sheaves of lower gerbes. For $p = 0$ this is trivial, simply stating that 0-gerbes are sheaves whose sections are continuous functions. For $p = 1$ we can consider the example given in [3] of the gerbe of spin structures on a space which admits a global SO structure but not a $Spin_c$ structure. In such a case it is natural to cover the space with open sets, on each of which it is possible to define a $Spin_c$ structure, and define transition functions between the structures. Since each structure is itself a line bundle (specifically, an S^1-bundle) this

construction is a 1-gerbe whose sections are local $Spin_c$ structures. The cohomology group H^{p+2} associated with this gerbe is essentially the mod 2 reduction of the second Steifel-Whitney class $w_2(P)$ (where P is the SO bundle) whose triviality implies that a $Spin_c$ structure can be globally defined. In this case (in the language of theorem 3.4) the gerbe would be topologically trivial and so has a global section, in this case the 0-gerbe (bundle) of $Spin_c$ structure.

We can take this construction slightly farther by noting that $G_p(X, K)$ forms a group. This can be shown by induction. The statement is clearly true for $p = -1$, using pointwise multiplication. Now if it is proven for some p, then an element of $G_{p+1}(X, K)$ is a sheaf of groups. We define the product of two sheaves by the pointwise multiplication of sections; i.e., if $\xi = (U_\alpha, s_\alpha)$ and $\xi' = (U'_\alpha, s'_\alpha)$, then $\xi\xi' = (U \cap U', (ss')_\alpha)$. The transition functions for this sheaf are $\phi_{\alpha\beta} = s_\alpha s'_\alpha s'^{-1}_\beta s^{-1}_\beta$. This clearly satisfies the group axioms, with the trivial sheaf acting as identity. Therefore an element of G_p is actually a bundle with sections in G_{p-1}, and so structural group G_{p-2}. Thus

Corollary 3.5: $G_p(X, K) \cong G_0(X, G_{p-2}(X, K))$.

This generalizes the theorem [3, 14] that Abelian 1-gerbes can be described as bundles of bundles. While we have used the Abelian property in deriving this result, we believe that a very similar result should hold in the non-Abelian case. In such a case, we know by this argument that p-gerbes are sheaves of equivalence classes of $(p-1)$-pregerbes, but not that these themselves form $(p-1)$-gerbes. The definition of a section structure continues to hold in this case. In both cases, it is clear that the existence of a global section of the sheaf (some $U_\alpha = X$) is equivalent to the existence of such for the gerbe to which it is associated, and so the conditions for their topological triviality must be equivalent. This is essentially the result used in the proof of proposition 2.2; we will develop it in a slightly more detailed form in section 4 as well.

We can also relate the sheaf picture to the de Rham picture. The connection on the sheaf associated to a p-gerbe is the G_{p-2}-valued one-form $\delta \log f^{(\alpha)}$, which by construction is equal to the one-form $A^{(1)}$ defined earlier. If we transform the Čech indices of $f^{(\alpha)}_{\beta_1 \cdots \beta_{p+1}}$ to spacetime indices as before, the sheaf connection is then equal to the gerbe connection, with one index of the gerbe connection corresponding to the one-form index of the sheaf connection, and the rest interpretable as internal indices. The relation to the loop picture is less clear, but can be found by going through the

de Rham construction.

4 K–theory of gerbes

The fact that gerbes are also sheaves suggests that they should have a natural K-theory. A natural choice is to define the K-theory of gerbes to simply be that of the associated sheaves; we will show that this is the same K-theory as one would get by directly defining the Whitney sum of gerbes. This K-theory then will classify sources of B-field charge (for example) in the same way that the usual K-theory of bundles classifies 1-form charges.

So in this section we will do the following: First, we will show that the gerbe Whitney sum agrees with the Whitney sum of sheaves related to the gerbes. We will use this to define the "topological" K-theory of gerbes (analogous to the topological K-theory of bundles) and show that it behaves like one would expect a K-theory to behave. We will then demonstrate the analogue of the Serre-Swan theorem, which for bundles relates their topological K-theory to the algebraic K-theory of the ring $C(X, K)$, and for gerbes allows us to relate this topological K-theory to an algebraic K-theory of $\mathfrak{alg}\, G_{p-2}$. This will give us the second recursion relation, which will allow us to make explicit calculations of K^0.

We begin with the Whitney sum of sheaves. To each p-gerbe ξ is associated a sheaf whose sections are the $f^{(\alpha)}$. The Whitney sum of the two sheaves associated to ξ and ξ' then has sections $f^{(\alpha)} \oplus f'^{(\alpha)}$ on each set in their mutual refinement. The addition $\oplus$ is simply the direct sum of two K-representations. By (14), this means that the "sheaflike" Whitney sum of two gerbes is another gerbe, with transition functions

$$g_{\alpha\beta_1\cdots\beta_{p+1}} \oplus g'_{\alpha\beta_1\cdots\beta_{p+1}} \ . \tag{20}$$

This is precisely what we would naturally define as the Whitney sum of two gerbes in the absence of any notion of associated sheaves. Therefore we can refer to this addition as the Whitney sum of gerbes without any hesitation.

Since this sum is a Whitney sum of sheaves, though, Swan's theorem applies, so that for every ξ there is a ξ' such that $\xi \oplus \xi'$ is trivial. (In the sheaf sense, that is that its class in $H^2(X, G_{p-2})$ is trivial) This triviality means that the sheaf associated to the sum posesses a global section; but this implies that the gerbe itself has a global section, and so $\xi \oplus \xi'$ is trivial in the gerbe sense as well. (That is, its cohomology in $H^{p+2}(X, K)$ is trivial) This proves that

Lemma 4.1: (Swan's Theorem for gerbes) For every $\xi \in G_p(X,K)$, there exists a $\xi' \in G_p(X,K)$ such that $\xi \oplus \xi'$ is trivial.

The Whitney sum therefore gives the set of homomorphism classes of p-gerbes the structure of a monoid, just as it does for sheaves. We can therefore define the K-group of p-gerbes $K^0[G_p(X,K)]$ to be the enveloping (Grothendieck) group of this monoid. By the relationship of Whitney sums of gerbes to the sums of the associated sheaves, the K-group of gerbes is equal to the K-group of sheaves, so K^0 commutes with the isomorphism of Corollary 3.5; i.e.,

Lemma 4.2: $K^0[G_p(X,K)] = K^0[G_0(X, G_{p-2}(X,K))]$.

This allows us to calculate K-groups of gerbes using the technology already developed for calculating the same groups for sheaves. It also means that the usual theorems of K-theory – in particular, the exact sequences and Bott periodicity – continue to apply to the K-theory of gerbes.

There is one particular theorem which it is worth examining in this case, namely the Serre-Swan theorem. This theorem ordinarily states that the topological K-theory of fiber bundles (the construction described above for $p=0$) is isomorphic to the algebraic K-theory K_0 of the module Γ of sections of bundles,[5] i.e.

$$K^0[G_0(X,K)] \cong K_0[\Gamma[G_0(X,K)]] = K_0[C(X,K)] \ . \tag{21}$$

Using propositon 3.4, this implies that

Proposition 4.3: $K^0[G_p(X,K)] \cong K_0[\Gamma[G_p(X,K)]] = K_0[G_{p-1}(X,K)]$, where the quantity on the left-hand side is the topological K-theory of gerbes defined above, and the quantity on the right-hand side is the algebraic K-theory of the group of $(p-1)$-gerbes defined in section 3. This and corollary 3.5 are our recursion relations. They can naturally be used to compute K-groups; for instance,

$$K^0[G_1(X,K)] \quad = \quad K^0[G_0(X, G_{-1}(X,K))]$$

[5]The K-theory of a $C^\star$-algebra such as $C(X,K)$ is defined (for algebras posessing a unit) to be the enveloping group of the monoid of homomorphism classes of projection operators in the algebra under a Whitney sum. This algebraic K-theory generalizes the ordinary topological K-theory of bundles, which is algebraically the K-theory of commutative algebras. The problem of non-unital $C^\star$-algebras is analogous to that of bundles on noncompact spaces, and is resolved by taking a unital extension of the algebra and then modding out its contribution to the K-group. The analogous procedure for topological K-theory is to move to the one-point compactification of the space, e.g. $K^0(\mathbb{R}^n) \equiv K^0(S^n)$. An accessible introduction to algebraic K-theory is given in [21].

$$
\begin{aligned}
&= K_0[G_{-1}(X, C(X,K))] \\
&= K_0[C(X, C(X,K))] \\
&= K_0[C(X^2, K)] \ .
\end{aligned}
\tag{22}
$$

This can be used along with the ordinary Sen construction [15]-[20] to determine the allowable types of NS B-field charge for branes in type II string theory. The process works identically to the K-theory classification of 1-form charges, [7, 8, 9] now using 1-gerbes and their Whitney sums. In type IIB, one describes p-branes as defects in D9-D$\bar{9}$ pairs. Then as for bundles, (all of the same reasoning applies) the B-field charge of a p-brane takes values in

$$
K^0[G_1(\mathbb{R}^{10-p}, \mathbb{C})] = K_0[C(\mathbb{R}^{20-2p}, \mathbb{C})] = \mathbb{Z} \ , \tag{23}
$$

where the latter is a standard result of topological K-theory.

In type IIA, there is the additional subtlety that 9-branes are not stable and so the simplest version of the Sen construction does not suffice. In this case, analogy with the bundle case suggests that the solution is to take a higher K-group K^{-1}. This group is defined for bundles as the group of pairs (E, α), where E is a bundle and α is an automorphism of E, with addition rule $(E,\alpha)+(F,\beta) = (E\oplus F, \alpha\oplus\beta)$ and modulo the equivalence $(E,\alpha) \sim (F,\beta)$ if there exist (E',α') and (F',β') such that α' and β' are homomorphic to the identity automorphism and $(E,\alpha)+(E',\alpha') \cong (F,\beta)+(F',\beta')$. We take the same definition for gerbes. As for K^0, this definition is insensitive to whether we use the gerbe or the sheaf Whitney sum, and the equivalent of lemma 4.2 applies as well. Since this is equivalent to a K-group of sheaves, the analogue of proposition 4.3 is valid as well; in this case, it is

$$
K^{-1}[G_p(X,K)] \cong K_1[\Gamma[G_p(X,K)]] = K_1[G_{p-1}(X,K)] \ . \tag{24}
$$

The group K_1 is another algebraic K-group.

We will not attempt to show whether K^{-1} classifies gerbe charges in type IIA theory as it does for bundles, but this is a reasonable expectation. If indeed it does, then we may calculate

$$
\begin{aligned}
K^{-1}[G_1(X,\mathbb{C})] &= K^{-1}[G_0(X, C(X,\mathbb{C}))] \\
&= K_1[C(X^2,\mathbb{C})] \\
&= K^{-1}[G_0(X^2,\mathbb{C})]
\end{aligned}
$$

By Bott periodicity, this is equal to $K^0[G_0(SX^2, \mathbb{C})]$, where the suspension SY of a manifold Y is defined to be the one-point compactification of $Y \times [0,1]$. In particular, for $X = \mathbb{R}^{10-p}$, $SX^2 = S^{21-2p}$, and so the B-field takes values in $K^0[G_0(S^{21-2p}, \mathbb{C})] = \mathbb{Z}$.

Therefore in both type II theories, all branes (both stable and unstable) can carry an integral NS B-field charge. This is not surprising since all such branes couple to the fundamental string, but is a good check on our picture of gerbes.

One should note that by construction, the modules of gerbes are commutative, and so in a sense all of the K-groups one finds for gerbes are the same as those found for bundles. But this should come as no surprise, since as we have seen gerbes are themselves very special bundles.

Gerbes are therefore prone to arise under a wide variety of circumstances. By theorem 2.1, any higher-form connection (a higher-form field with appropriate gauge symmetries and transition functions, or equivalently a higher-form field strength) leads to a gerbe; by theorem 3.4, whenever there is a topological obstruction to forming an (Abelian) bundle we have a gerbe. Finally, by theorem 2.3 any gauge symmetry where the transforming objects are extended objects is naturally described by a gerbe.

It is interesting that the converses of these statements are true as well. In particular, the presence of a higher-form connection implies a gauge symmetry realized on extended objects in the theory. Consider, for example, the case $p = 1$, where we have a 2-form $B^{\mu\nu}$ with field strength $H = \nabla B$. Let us restrict our attention to the case of K abelian so that we need not concern ourselves with the distinction between ∇ and d. Then our action is likely to contain terms such as $H \wedge \star H$ (for a Yang-Mills-like theory) or $B \wedge H$. (For a Chern-Simons-like theory) Using (5), we can write at least formally $B = \delta^{-1} dA$, where A is the $\mathfrak{alg}$ K-valued 1-form connection. The inverse coboundary operator δ^{-1} is clearly nonlocal; it effectively integrates over a 1-cochain. Therefore we can consider our connection B to be the integral of a 2-form over a 1-dimensional internal space. This is of course consistent with our loop picture, since B is a connection over M_1X.

For $p = 1$, it is also straightforward to evaluate M_1X; by definition, it is the space of submanifolds homomorphic to dimension-1 submanifolds which are not homomorphic to dimension-0 submanifolds, i.e. points, and so M_1X is simply the space of submanifolds homomorphic to loops in X. Therefore this theory may be reinter-

preted as a theory of 1-forms taking values in $\mathfrak{alg}\ K \times \Omega^1 X$, and the extended objects found in our theory are closed strings. Similar arguments can be made for higher p; for example, for $p = 2$ and X compact, $M_2 X$ is the space of (submanifolds homomorphic to) Riemann surfaces in X, while for X noncompact $M_2 X$ also includes the family of infinite membranes. This result agrees with the known relationship of p-forms and extended objects in M-theory.

It is therefore natural to consider p-gerbes to be the generalization of bundles relevant to theories which have higher forms and extended objects. Using the geometric constructions and the K-theory defined above, these can be treated on a reasonable physical footing; they posess conserved charges, covariant field strengths, and gauge symmetries. However, several important issues, notably the definition of the lower-form connections and the sheaf picture in the non-Abelian case, still must be resolved.

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